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Renormalization group equations for effective field theories

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Abstract. We derive the renormalization group equations for a generic non-renormalizable theory. We show that the equations allow one to derive the structure of the leading divergences at any loop order in terms of one-loop diagrams only. In chiral perturbation theory, e.g., this means that one can obtain the series of leading chiral logs by calculating only one-loop diagrams. We discuss also the renormalization group equations for the subleading divergences, and the crucial role of counterterms that vanish at the equations of motion. Finally, we show that the renormalization group equations obtained here apply equally well also to renormalizable theories.

1 Introduction

Quantum field theories (QFT) which are used in phenomenology are tested up to a limited level of precision and in limited ranges of energies. In their formulation and application one does not need to worry about if and how the theory has to be modified once certain boundaries in energy (or precision) are crossed: in such cases one usually speaks of effective field theories. The property of renormalizability of such quantum field theories is conceptually not particularly relevant[1] – at most one can work out predictions to an interesting level of precision using only the renormalizable part of the interaction Lagrangian. The latter is the case of the standard model, where the level of precision which has been reached without need for new non-renormalizable interactions has become surprisingly high.

In renormalizable quantum field theories one of the most useful tools is that of the renormalization group equations (RGE). After the renormalization procedure, the coupling constants which define the theory acquire a dependence on an arbitrary energy scale. It is convenient to identify the latter with the typical energy scale of the process under consideration – the strength of the interaction then varies with the energy at which this occurs. The RGE dictate how the coupling constants depend on the scale, and are one of the most important intrinsic properties of a quantum field theory. As is well known, the discovery of the property of asymptotic freedom for non-abelian gauge theories was a major breakthrough and showed that such theories could be candidates for describing the observed behavior of hadrons in deep inelastic scattering, which then led to the formulation of QCD.

[1] This is the point of view of many modern textbooks on quantum field theories. See, e.g., [1, 2]

The use of the RGE in non-renormalizable QFT has not received the same attention, and has not yet been studied thoroughly. On the one hand this may be due to the different use of non-renormalizable QFT, where one usually does not have the problem of evolving coupling constants over order of magnitudes in energy scales. On the other hand, the very structure of the RGE in the case of non-renormalizable QFT is a lot more complicated than for renormalizable ones. One of the very first investigations of this issue was made by Weinberg [3], in his seminal paper on the effective Lagrangians. There he shows how one can follow the same reasoning that leads to the RGE for renormalizable theories to obtain information about the structure of the two-loop divergences in chiral perturbation theory (CHPT). He does not attempt, however, to push the analysis to higher orders. This is the aim of the present paper.

What is the physical information one would like to obtain from such an analysis? To illustrate the answer let us consider, for example, the expansion of the pion mass in quark masses [6]:

$$M_\pi^2 = M^2\left[1+\frac{M^2}{(4\pi F)^2}\left(-\frac{1}{2}\log\frac{M^2}{\mu^2}+\ell_3^r(\mu)\right)\right. \tag{1.1}$$
$$\left.+\frac{M^4}{(4\pi F)^4}\left(\frac{17}{8}\log^2\frac{M^2}{\mu^2}+\dots\right)+O(M^6)\right],$$

where $M^2 = 2\langle\bar{q}q\rangle\hat{m}/F^2$ is the Gell-Mann–Oakes–Renner term, and F the pion decay constant in the chiral limit. We have stopped the expansion at the next-to-next-to-leading order, and at this order have written down explicitly only the double chiral logarithm. Indeed, what we want to show here is that the coefficient of the single (double) chiral logarithm at order M^4 (M^6) is a pure number, and does

not involve any of the new coupling constants that show up at each order in the chiral expansion. It follows from simple power counting that this remains true to all orders: the coefficient of the leading chiral log at any order in the chiral expansion is a pure number. The analysis of Weinberg [3] concerned precisely the coefficient of the double chiral log: he showed that, although in principle that coefficient is the sum of contributions of several different loop diagrams, its value is constrained by the RGE, and can in fact be obtained from one-loop diagrams only [6]. Since the leading chiral log is potentially the dominating correction at each chiral order (this statement is of course μ-dependent, and typically valid for values of μ around 1 GeV), the RGE do provide information of phenomenological interest. If one were able to sum the whole series of the leading chiral logs with the help of the RGE, this would certainly be a very useful and exciting result.

Analogous statements are true for any observables, and indeed one can obtain the complete expression of the double chiral logs from the RGE in the generating functional [8], in the formulation of CHPT with external fields which is due to Gasser and Leutwyler [4]. The validity of the RGE at the two-loop level has then been explicitly verified in [9] in the full two-loop calculations of the divergence structure of CHPT. The extension of these RGE arguments to higher orders has however not yet been made in the framework of CHPT.

The RGE in a non-renormalizable QFT have been studied for the case of the non-linear σ-model in two dimensions [14, 15]. This low-dimensional QFT is particularly interesting because, on the one hand, if one does not specify the metric of the manifold on which the fields live, it is non-renormalizable. But on the other hand, the structure of the possible counterterms is severely constrained: only two space-time derivatives of the fields can appear, such that the counterterms can always be absorbed in a redefinition of the metric. As was shown by Friedan [14] one can write down the RGE for the metric, which do imply interesting constraints on the form of the leading divergences at higher orders in the loop expansion. This was further analyzed and clarified by Alvarez-Gaumé, Freedman and Mukhi [15], who showed that on the basis of the RGE one can derive the leading two-loop divergences from purely geometrical considerations (the Palatini identity). They then verified that the actual two-loop calculation gave results in agreement with the RGE. In that paper the RGE for the leading divergences were derived to all orders (although in a rather implicit form). A few years later, Kazakov [11] extended these ideas to arbitrary QFT in four space-time dimensions. He relied, however, on a specific assumption on the scaling of the Lagrangian with the renormalization scale μ (the RGE were derived in dimensional regularization), and the RGE were also given in a very implicit form, completely analogous to those for the metric in the 2-dimensional non-linear σ-model. As we will show later, however, this scaling cannot hold in CHPT, and in our analysis we have had to adopt a different starting point.

The structure of this paper is as follows: in Sect. 2 we set our notation and derive the implicit form of the RGE. In Sect. 3 we analyze the RGE for the leading divergences, first order by order in the loop expansion, and then give the explicit all-order formula and discuss its meaning. In Sect. 4 we consider the RGE for the subleading divergences, and discuss them for the first few orders in the loop expansion. In Sect. 5 we discuss the role of the counterterms that vanish at the solution of the equations of motion, in connection with the role of one-particle-reducible graphs. In Sect. 6 we show that the RGE we have derived apply equally well to the case of a renormalizable QFT, and explicitly discuss the case of the $O(N)$ invariant ϕ^4 theory. In that framework we can also illustrate the role of the counterterms that vanish at the solution of the equations of motion. In Sect. 7 we add a ϕ^6 interaction to the ϕ^4 theory and analyze how the RGE are affected. Finally we summarize our results in Sect. 8. In the appendices we discuss the more technical points, and in particular the derivation of the RGE to all orders.

2 Renormalization group equations

2.1 Notation

A quantum field theory is defined by specifying its classical action S_0 and a series of quantum corrections S_i:

$$S[\phi, J] = \sum_{n=0}^{\infty} \hbar^n S_n[\phi, J]\,. \tag{2.1}$$

Each term in the series is a function of a number of fields, collectively denoted with the symbol ϕ, and of external sources J coupled to operators O (which can be either fields ϕ or functions thereof). By evaluating the path integral

$$\mathrm{e}^{\mathrm{i}Z[J]/\hbar} := \frac{1}{\mathcal{N}} \int \prod [\mathrm{d}\phi_i]\, \mathrm{e}^{\mathrm{i}S[\phi,J]/\hbar}\,, \tag{2.2}$$

one obtains the generating functional $Z[J]$ of all connected Green functions of the operators O_J as a power series in $\hbar$,

$$Z[J] = \sum_{n=0}^{\infty} \hbar^n Z_n[J]\,. \tag{2.3}$$

In the evaluation of the path integral divergences are generated: these need to be renormalized in order to have physically meaningful results. This can be done in the following way. The action S is the integral over spacetime of the bare Lagrangian which also admits an expansion in a power series in $\hbar$

$$S[\phi, J] = \int \mathrm{d}x \mathcal{L}^{\mathrm{bare}}(\phi, J)\,, \qquad \mathcal{L}^{\mathrm{bare}} = \sum_{n=0}^{\infty} \hbar^n \mathcal{L}^{(n)\mathrm{bare}}\,. \tag{2.4}$$

We regularize the theory by working in d spacetime dimension, and split the bare Lagrangians into a renormalized and a divergent part:

$$\mathcal{L}^{(n)\mathrm{bare}} := \mu^{-\varepsilon n}(\mathcal{L}^{(n)} + \mathcal{L}^{(n)\mathrm{div}}) \quad n \geq 0\,, \tag{2.5}$$

where $\mathcal{L}^{(n)}$ is the renormalized Lagrangian, $\varepsilon := 4-d$, and $\mathcal{L}^{(n)\mathrm{div}}$ $n \geq 1$ diverges in the limit $\varepsilon \to 0$. All the divergences generated in the calculation of the path integrals are local (see, e.g. [1]), and can be reabsorbed by properly defining $\mathcal{L}^{(n)\mathrm{div}}$.

The scale μ introduced in (2.5) serves the purpose of having a renormalized Lagrangian of dimension d for all $\hbar$-orders. The reason why this choice is the correct one in CHPT is explained in Appendix A. In case of a renormalizable Lagrangian other choices would be more appropriate – on the other hand, the physical content of the RGE does not depend on this, as we will see in Sect. 6. From now on we set $\hbar = 1$. In the framework of CHPT the renormalized Lagrangian $\mathcal{L}_n$ corresponds to the Lagrangian of chiral order $2n+2$.

The divergent part of the bare Lagrangian of $\hbar$-order n can be written as a sum of poles in ε:

$$\mathcal{L}^{(n)\mathrm{div}} := \sum_{k=1}^{n} A_k^{(n)} \varepsilon^{-k} = \sum_{k=1}^{n} \sum_{l=k}^{n} \mathcal{A}_{lk}^{(n)} \varepsilon^{-k}\,, \tag{2.6}$$

where after the second equality sign we have expanded the divergences in terms generated by diagrams with l loops – obviously a term diverging like ε^{-k} can only be generated by diagrams with at least k loops. The part of $\hbar$-order n of the bare Lagrangian therefore reads

$$\mathcal{L}^{(n)\mathrm{bare}} = \mu^{-\varepsilon n}\left[\mathcal{L}^{(n)} + \sum_{k=1}^{n} A_k^{(n)} \varepsilon^{-k}\right]. \tag{2.7}$$

The calculation of the divergent coefficients $A_k^{(n)}$ can be performed in various different ways, which we need not specify here. The use of the background field method and the heat-kernel techniques are particularly convenient in cases where a local symmetry is present, like for gauge theories or CHPT. Concrete examples of calculations of $A_k^{(n)}$ up to $n = 2$ for gauge theories and CHPT can be found in [9,10], respectively.

At each $\hbar$-order both the Lagrangian $\mathcal{L}^{(n)}$ and the pole coefficients $A_k^{(n)}$ can be expanded in a minimal basis of operators $\mathcal{O}_i^{(n)}$, $i = 1, \ldots, M_n$:

$$\begin{aligned}
\mathcal{L}^{(n)} &= \sum_{i=1}^{M_n} c_i^{(n)} \mathcal{O}_i^{(n)} = \vec{c}^{\,(n)} \cdot \vec{\mathcal{O}}^{(n)}\,,\\
A_k^{(n)} &= \sum_{i=1}^{M_n} a_{k\,i}^{(n)} \mathcal{O}_i^{(n)} = \vec{a}_k^{\,(n)} \cdot \vec{\mathcal{O}}^{(n)}\\
&= \sum_{l=k}^{n} \vec{a}_{lk}^{\,(n)} \cdot \vec{\mathcal{O}}^{(n)}\,.
\end{aligned} \tag{2.8}$$

For a renormalizable theory M_n is a constant independent of n, and the minimal basis of operators is the same for every n, whereas for a non-renormalizable one M_n is a growing function of n. In the present formalism this is the only difference between a renormalizable and a non-renormalizable theory.

2.2 Renormalization group equations

The RGE follow from the requirement that the bare Lagrangians[2] (2.7) do not depend on the scale μ:

$$\begin{aligned}
0 &= \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \mathcal{L}^{(n)\mathrm{bare}}\\
&= \mu^{-\varepsilon n}\left\{-\varepsilon n\left[\mathcal{L}^{(n)} + \sum_{k=1}^{n} \varepsilon^{-k} A_k^{(n)}\right]\right.\\
&\qquad \left. + \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \mathcal{L}^{(n)} + \sum_{k=1}^{n} \varepsilon^{-k} \mu \frac{\mathrm{d}}{\mathrm{d}\mu} A_k^{(n)}\right\}.
\end{aligned} \tag{2.9}$$

The μ-dependence of the $\mathcal{L}^{(n)}$ will be described by its β-function, denoted by the symbol $\mathcal{B}$:

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \mathcal{L}^{(n)} = \mathcal{B}^{(n)} + \varepsilon n \mathcal{L}^{(n)}\,, \tag{2.10}$$

where the ε-dependence has been explicitly subtracted. The β-function of a Lagrangian is also expandable in the set of operators:

$$\mathcal{B}^{(n)} = \sum_{i=1}^{M_n} \beta_i^{(n)} \mathcal{O}_i^{(n)} = \vec{\beta}^{\,(n)} \cdot \vec{\mathcal{O}}^{(n)}\,, \tag{2.11}$$

and is defined to be evaluated at $d = 4$. If we decompose (2.10) into a basis of operators we obtain for each of the coupling constants $c_i^{(n)}$

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} c_i^{(n)} = \beta_i^{(n)} + \varepsilon n c_i^{(n)}\,. \tag{2.12}$$

The μ-dependence of the divergent parts $A_k^{(n)}$ can only arise through their explicit polynomial dependence on the coupling constants $c_i^{(n)}$. We can therefore rewrite the μ derivative as

$$\begin{aligned}
\mu \frac{\mathrm{d}}{\mathrm{d}\mu} &= \sum_n \mu \frac{\mathrm{d}\vec{c}^{\,(n)}}{\mathrm{d}\mu} \cdot \vec{\partial}^{(n)} = \sum_n \left[\vec{\beta}^{\,(n)} + \varepsilon n \vec{c}^{\,(n)}\right] \cdot \vec{\partial}^{(n)}\\
&= \nabla + \varepsilon N_v\,,
\end{aligned} \tag{2.13}$$

where $\vec{\partial}^{(n)} := \partial / \partial \vec{c}^{\,(n)}$, and we have introduced the following definitions:

$$\begin{aligned}
\nabla &:= \sum_n \nabla^{(n)}\,, & \nabla^{(n)} &:= \vec{\beta}^{\,(n)} \cdot \vec{\partial}^{(n)}\,,\\
N_v &:= \sum_n n D^{(n)}\,, & D^{(n)} &:= \vec{c}^{\,(n)} \cdot \vec{\partial}^{(n)}\,.
\end{aligned} \tag{2.14}$$

The operator N_v and its eigenvalues have a clear meaning which we are now going to illustrate. We saw above that every Lagrangian $\mathcal{L}^{(j)}$ comes with a factor $\hbar^j$. Consider a generic object on which the operator N_v will act, e.g. an $A_k^{(n)}$, and more in particular the contribution of a

[2] From now on we suppress the superscript r on the renormalized Lagrangian

specific loop graph to it, denoted by $\mathcal{G}$. When acting on a diagram, the operator $D^{(j)}$ will yield the number n_j of vertices coming from the Lagrangian $\mathcal{L}^{(j)}$ which are present in that diagram:

$$D^{(j)}\mathcal{G} := n_j\mathcal{G}\,. \tag{2.15}$$

For N_v we therefore get

$$N_v\mathcal{G} = \sum jn_j\mathcal{G} =: n_v\mathcal{G}\,, \tag{2.16}$$

where n_v (called v-order) is the contribution to the $\hbar$-order of a diagram $\mathcal{G}$ which is coming only from the vertices. The total $\hbar$-order of the diagram must be larger or equal to n_v, and the difference between n and n_v is generated dynamically by the n_l loops present in that diagram: $n = n_v + n_l$ (for n_l we will use the term l-order). As already mentioned above, for a renormalizable theory we would have $n_v \equiv 0$ and therefore $n = n_l$.

We can now write (2.9) in a very compact form:

$$\mathcal{B}^{(n)} = N_l A_1^{(n)}\,, \tag{2.17}$$

$$N_l A_{k+1}^{(n)} = \nabla A_k^{(n)}\,, \qquad k = 1,\dots,n-1\,, \tag{2.18}$$

where N_l is the operator that yields the l-order of the object it acts on, being defined by

$$N_l := n - N_v\,. \tag{2.19}$$

We observe that terms of different l-order in (2.17) and (2.18) cannot mix with each other. One way to prove this statement is the following: all the objects appearing in (2.17) and (2.18) are polynomials in the coupling constants $c_i^{(n)}$. Since these identities hold no matter what the value of these constants is, they must hold for the coefficients of each monomial in the coupling constants. One can now group together all monomials with the same n_v, which also have the same l-order $l = n - n_v$. This follows from the fact that the $\hbar$-order n is constant for all terms in (2.17) and (2.18).

The RGE can therefore be decomposed into sets of equations with fixed l-order

$$\mathcal{B}_l^{(n)} = l\mathcal{A}_{l1}^{(n)}\,, \quad l = 1,\dots n\,, \tag{2.20}$$

$$l\mathcal{A}_{lk}^{(n)} = \sum_{l'=1}^{l-k+1} \nabla_{l'}\mathcal{A}_{l-l'\,k-1}^{(n)}\,, \quad l = k,\dots n,\ k = 2,\dots,n\,, \tag{2.21}$$

where the additional index l stands for the loop order, and where ∇_l is defined by

$$\nabla_l := \sum_{n=l}^{\infty} \vec{\beta}_l^{(n)} \cdot \vec{\partial}^{(n)}\,. \tag{2.22}$$

The boundaries in the sum follow from the trivial observation that $A_k^{(n)}$ has l-order $\geq k$ – the $\hbar$-order is of course equal to n.

In order to further manipulate the RGE it is useful to establish the following simple rules.

(1) $\mathcal{B}^{(n)}$ and $\mathcal{A}^{(n)}$ can carry any l-order and v-order which add up to n. The $c_i^{(n)}$ have by definition $N_l c_i^{(n)} = 0$, $N_v c_i^{(n)} = n$. A derivative $\partial_i^{(n)}$ reduces the v-order of the object it acts on by n.
(2) With the action of ∇_l we differentiate with $\vec{\partial}^{(n)}$ and multiply the result with the corresponding $\vec{\beta}_l^{(n)}$. The net change in the $\hbar$-order n is therefore zero: ∇_l increases (decreases) the l-order (v-order) of the object it acts on by l:

$$N_l(\nabla_{l_1}\mathcal{A}_{l'k}^{(n)}) = (l_1 + l')\nabla_{l_1}\mathcal{A}_{l'k}^{(n)}\,,$$
$$N_v(\nabla_{l_1}\mathcal{A}_{l'k}^{(n)}) = (n - l_1 - l')\nabla_{l_1}\mathcal{A}_{l_2k}^{(n)}\,.$$

(3) If we have not enough $c_i^{(k)}$ inside of $\mathcal{A}_{l'k}^{(n)}$ on which we can act with the derivatives $\vec{\partial}^{(n)}$, the $\nabla_l\mathcal{A}_{l'k}^{(n)}$ will evaluate to zero:

$$\nabla_l\mathcal{A}_{l'k}^{(n)} = 0\,; \quad \forall l + l' > n\,.$$

The above statements are valid also for products of ∇_l if we substitute

$$\nabla_l \to \nabla_{l_1}\nabla_{l_2}\dots\nabla_{l_k}\,, \qquad l \to l_1 + l_2 + \dots + l_k\,. \tag{2.23}$$

3 RGE for the highest poles

In this section we analyze in detail the RGE (2.21) for the highest poles (HPRGE) $k = n$ and write them in a more compact form. Before getting to the final result for generic n, we find it instructive to examine a first few explicit cases starting from $n = 1$.

3.1 RGE for $n = 1,\ 2$ and 3

At lowest $\hbar$-order the RGE are practically trivial:

$$\mathcal{B}_1^{(1)} = \mathcal{A}_{11}^{(1)}\,, \tag{3.1}$$

and they only state that the scale dependence of the couplings in the $\mathcal{L}_1$ Lagrangian is determined by the one-loop divergences [3–5].

At the two loop level the equations become more interesting, as has been already observed by Weinberg [3] and others [6,8,9]:

$$\mathcal{B}_1^{(2)} = \mathcal{A}_{11}^{(2)}\,,$$
$$\mathcal{B}_2^{(2)} = 2\mathcal{A}_{21}^{(2)}\,,$$
$$2\mathcal{A}_{22}^{(2)} = \nabla_1\mathcal{A}_{11}^{(2)} = \nabla_1\mathcal{B}_1^{(2)}\,. \tag{3.2}$$

In this case the RGE show that the scale dependence of the couplings in $\mathcal{L}_2$ is fully contained in the single pole in ε, $\mathcal{A}_{21}^{(2)}$. Acting with N_l on $A_1^{(2)}$ yields the two terms $\mathcal{A}_{11}^{(2)} + 2\mathcal{A}_{21}^{(2)}$, where the first will be linear in the couplings coming from $\mathcal{L}^{(1)}$, and the latter will depend only on $\mathcal{L}^{(0)}$.

Fig. 1. Graphical representation of the RGE for $n = 2$

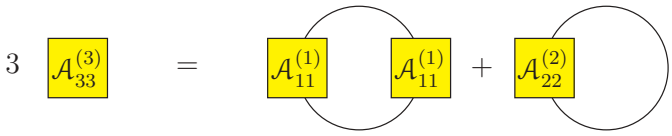

Fig. 2. Graphical representation of the RGE for $n = 3$

The first of these equations is identical to (4.5) in [8], whereas the second is equal to (4.6) or (2.44) also in [8]. As has already been observed [3,6,8] the second of these equations allows one to calculate the double chiral logs only with one-loop calculations. In passing we note that $\nabla_1^{(1)}$ can also be written as

$$\nabla_1^{(1)} = \vec{a}_{11}^{(1)} \cdot \vec{\partial}^{(1)} \,, \tag{3.3}$$

as follows from (3.1). For later convenience we introduce the symbol

$$d_n := \vec{a}_{nn}^{(n)} \cdot \vec{\partial}^{(n)} \,. \tag{3.4}$$

With this notation (3.3) can be re-expressed as $\nabla_1 = d_1$. We stress that all the d_n commute:

$$[d_n, d_m] = 0 \,. \tag{3.5}$$

At $n = 3$ we start the exploration of unknown territory – the RGE read

$$\begin{aligned}
\mathcal{B}_1^{(3)} &= \mathcal{A}_{11}^{(3)} \,, \\
\mathcal{B}_3^{(3)} &= 2\mathcal{A}_{21}^{(3)} \,, \\
\mathcal{B}_3^{(3)} &= 3\mathcal{A}_{31}^{(3)} \,, \\
3\mathcal{A}_{33}^{(3)} &= \nabla_1 \mathcal{A}_{22}^{(3)} \,, \\
2\mathcal{A}_{22}^{(3)} + 3\mathcal{A}_{32}^{(3)} &= (\nabla_1 + \nabla_2) \left[\mathcal{A}_{11}^{(3)} + \mathcal{A}_{21}^{(3)} \right] \,.
\end{aligned} \tag{3.6}$$

If we act with ∇_1 on the last equation, we can rewrite the next-to-last as

$$3!\mathcal{A}_{33}^{(3)} = \nabla_1^2 \mathcal{A}_{11}^{(3)} = \nabla_1^2 \mathcal{B}_1^{(3)} \,, \tag{3.7}$$

which shows again that all the information about the three-loop divergences is contained in the single pole in ε. The equation is however not yet fully explicit: the operator ∇_1 in fact contains derivatives $\vec{\partial}^{(n)}$ with all n, but obviously only the first few may contribute:

$$\nabla_1^2 \mathcal{B}_1^{(3)} = \left[\nabla_1^{(1)} \right]^2 \mathcal{B}_1^{(3)} + \nabla_1^{(1)} \nabla_1^{(2)} \mathcal{B}_1^{(3)} \,, \tag{3.8}$$

where in the last term the operator $\nabla_1^{(1)}$ acts only on $\vec{\beta}_1^{(2)}$ inside $\nabla_1^{(2)}$. From (3.2) we see that $\nabla_1^{(1)} \mathcal{B}_1^{(2)} = 2\mathcal{A}_{22}^{(2)}$, and therefore, that the HPRGE for $n = 3$ can be rewritten as

$$3!\mathcal{A}_{33}^{(3)} = \left(d_1^2 + 2d_2 \right) \mathcal{B}_1^{(3)} \,. \tag{3.9}$$

3.2 Highest-pole equation to all orders

After having analyzed explicitly the first few cases, it should now be clear how to extend the derivation of the same equation to all orders. The only rule that we need to use is rule (3) at the end of Sect. 2. The highest-pole equation for a generic n reads

$$n!\mathcal{A}_{nn}^{(n)} = \nabla_1^{n-1} \mathcal{A}_{11}^{(n)} = \nabla_1^{n-1} \mathcal{B}_1^{(n)} \,. \tag{3.10}$$

Such a simple expression is suggestive, but not very illuminating, because, as we have seen above for the $n = 3$ case, the product of the ∇_1 is a complicated object, due to the non-commuting property of the ∇_1. In the $n = 3$ case, however, we have seen that one can rewrite the product of two ∇_1s in terms of $d_{1,2}$ – objects which have a clear meaning and which commute between themselves. This fact indeed generalizes to all orders, and allows us to rewrite the product of $n-1$ ∇_1s (acting on $\mathcal{B}_1^{(n)}$) in terms of the d_k only with $k \leq n-1$, and to give a clear meaning to the HPRGE:

$$n\mathcal{A}_{nn}^{(n)} = \left[\sum_{\underline{\gamma}} \frac{1}{\prod_{k=1}^{l} \beta_k!} (d_{\alpha_1})^{\beta_1} \cdot \ldots \cdot (d_{\alpha_l})^{\beta_l} \right] \mathcal{B}_1^{(n)} \,, \tag{3.11}$$

where one sums over all $\underline{\gamma} := \{\alpha_i, \beta_i\}$ having the property $\sum_i \alpha_i \beta_i = n - 1$. The derivation of this formula can be found in Appendix B.

The effect of the operator d_i on any diagram is to substitute a vertex coming from the Lagrangian $\mathcal{L}^{(i)}$ with the corresponding highest-pole counterterm $\mathcal{A}_{ii}^{(i)}$. The formula (3.11) implies that the highest-pole counterterm is obtained by calculating all one-loop diagrams contributing to $\mathcal{B}_1^{(n)}$ and substituting *all* the vertices from the Lagrangians $\mathcal{L}^{(i)}$, $i < n$, with the corresponding highest-pole counterterm $\mathcal{A}_{ii}^{(i)}$.

Notice that all $\mathcal{A}_{ii}^{(i)}$ can be expressed in terms of $\mathcal{B}_1^{(j)}$s with $j \leq i$ only: the highest-pole counterterms can all be calculated with one-loop diagrams. On the other hand, (3.11) does not lend itself to an explicit direct evaluation to all orders, but can only be used recursively. Also, each step up in the recursive procedure is a non-trivial (although in principle straightforward) one-loop calculation.

4 RGE for the subleading poles

In the case of the subleading poles, the situation becomes somewhat more complicated. First of all we have to deal with two different possible l-orders for the terms with $k = n - 1$. The corresponding equations read

$$\begin{aligned}
(n-1)\mathcal{A}_{n-1\,n-1}^{(n)} &= \nabla_1 \mathcal{A}_{n-2\,n-2}^{(n)} \,, \\
n\mathcal{A}_{n\,n-1}^{(n)} &= \nabla_1 \mathcal{A}_{n-1\,n-2}^{(n)} + \nabla_2 \mathcal{A}_{n-2\,n-2}^{(n)} \,.
\end{aligned} \tag{4.1}$$

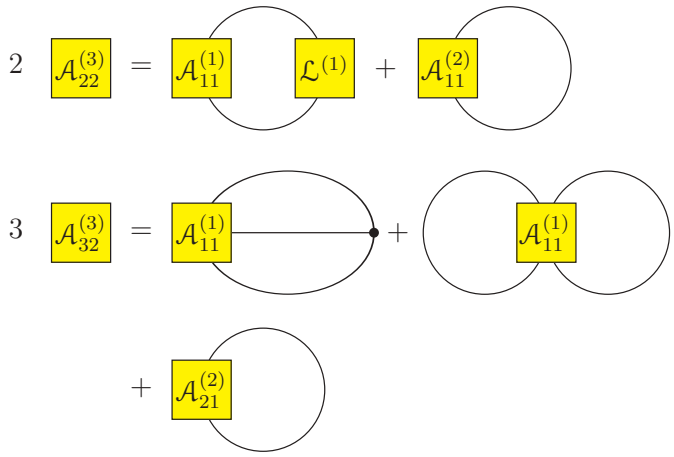

Fig. 3. Graphical representation of the RGE for the subleading poles ($n = 3$)

Just like for the highest-pole equation, we need to relate the left-hand side of this equation with the right-hand side of the equation with k one unit lower, until we reach $k = 1$. One can do this easily for the first equation in (4.1) – the result has exactly the same form as the one for the highest-pole equation:

$$(n-1)!\mathcal{A}^{(n)}_{n-1\,n-1} = \nabla_1^{n-2}\mathcal{A}^{(n)}_{11}\,. \tag{4.2}$$

In the second equation in (4.1) the right-hand side contains two terms, and each of them comes with a different coefficient in the next equation. The combinatorics is therefore somewhat more complicated, but can also be worked out without major difficulties. The result reads

$$\begin{aligned} &n!\mathcal{A}^{(n)}_{n\,n-1} \qquad\qquad (4.3)\\ &= 2\nabla_1^{n-2}\mathcal{A}^{(n)}_{21} + \sum_{j=0}^{n-3}(n-1-j)\nabla_1^j\nabla_2\nabla_1^{n-3-j}\mathcal{A}^{(n)}_{11}\,. \end{aligned}$$

An analysis of the above formula to all orders is provided in Appendix C: the result we obtain is similar to (3.11) for the highest poles, but admittedly considerably more involved. As in the case of the highest poles, also here the formula can only be used in a recursive manner: to obtain the subleading poles at $\hbar$-order n one has first to work out all leading and subleading poles of $\hbar$-order $n' < n$, and insert these as vertices in one- and two-loop diagrams. For this reason we provide the discussion of the formula to all orders only in the appendix and consider here the RGE for the subleading poles for the first few $\hbar$-orders.

4.1 RGE for the subleading poles for $n = 2$ and 3

The first subleading poles appear at $n = 2$. We have already seen the RGE for this case in the previous section, but we concentrated there on the leading pole. As for the subleading poles, at this $\hbar$-order the equations only relate these to the corresponding β:

$$\mathcal{B}^{(2)}_1 = \mathcal{A}^{(2)}_{11}\,,$$

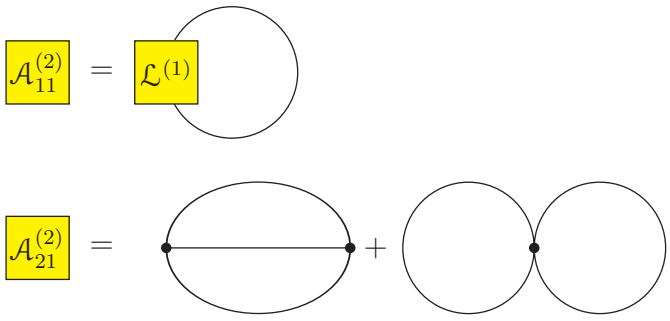

Fig. 4. Vertices needed for the calculation of the subleading poles ($n = 3$)

$$\mathcal{B}^{(2)}_2 = 2\mathcal{A}^{(2)}_{21}\,. \tag{4.4}$$

At $n = 3$ the RGE for the subleading poles start to provide some interesting information. All the RGE at $n = 3$ are given in (3.6) – here we consider only those relevant for the subleading poles, and split them according to the l-order, as in (4.1):

$$\begin{aligned} 2\mathcal{A}^{(3)}_{22} &= \nabla_1\mathcal{A}^{(3)}_{11} = \left(\nabla_1^{(1)} + \nabla_1^{(2)}\right)\mathcal{B}^{(3)}_1\,, \qquad (4.5)\\ 3\mathcal{A}^{(3)}_{32} &= \nabla_1\mathcal{A}^{(3)}_{21} + \nabla_2\mathcal{A}^{(3)}_{11} = \frac{1}{2}\nabla_1^{(1)}\mathcal{B}^{(3)}_2 + \nabla_2^{(2)}\mathcal{B}^{(3)}_1\,. \end{aligned}$$

The meaning of these equations is as follows: the first one implies that the part of the subleading poles that comes from two-loop diagrams is given by the single pole from one-loop diagrams ($\mathcal{A}^{(3)}_{11}$) after one has substituted one $\mathcal{L}^{(1)}$ ($\mathcal{L}^{(2)}$) vertex with $\mathcal{A}^{(1)}_{11}$ ($\mathcal{A}^{(2)}_{11}$). According to the second one the double pole coming from three-loop diagrams is given by two terms: the first one is obtained from single poles from two-loop diagrams after substitution of one $\mathcal{L}^{(1)}$ vertex with $\mathcal{A}^{(1)}_{11}$, and the second one from single poles from one-loop diagrams after substitution of one $\mathcal{L}^{(2)}$ vertex with $2\mathcal{A}^{(2)}_{21}$, i.e. the subleading poles of one $\hbar$-order lower.

5 Role of one-particle-reducible diagrams

In the graphical representations of the RGE we have always drawn one-particle-irreducible (1PI) graphs, although also one-particle-reducible (1PR) ones contribute to the generating functional Z, and possibly to its divergences. It would indeed be desirable to be in a setting where only 1PI graphs contribute to divergences and therefore have to be considered in the RGE. In this section we show how one can ensure that this setting is realized. The problem is related to the role of counterterms that vanish at the classical solution of the equations of motion (EoM terms in short). At each $\hbar$-order n all the divergences are absorbed by the counterterms $\mathcal{O}^{(n)}_i$, and we may assume that they form a minimal basis, i.e. that they are all linearly independent. In the reduction procedure of a complete list of counterterms to a minimal basis one eliminates terms which are algebraically linearly dependent (such as terms related by trace identities in CHPT), terms which are a

total derivative, and also the EoM terms. While the first two categories of terms can be eliminated without any consequences on the renormalization procedure, eliminating EoM terms is a less trivial issue, which has to be discussed in some detail. This problem has already been dealt with in [8,9,12], in performing the renormalization at the two-loop level. As observed (and explicitly verified) in [8], one can choose the coefficient in front of the EoM terms in such a way that the sum of 1PR graphs is finite. Here we discuss how this can be done to any loop order.

The sum of all 1PI graphs defines the generating functional of proper vertices (or effective action) Γ – we denote the sum of 1PR graphs with $Z^{\rm 1PR}$. In the background field method one shifts the fields over which the path integral is performed, $\phi \to \phi + \xi$, and then one integrates over the ξ fields. In this framework Γ, the sum of all 1PI diagrams, becomes a functional both of the fields ϕ (which need not be fixed at the solution of the EoM) and of the external sources J. Order by order in the loop expansion we have

$$Z_n[J] = \Gamma_n[\phi, J]_{|_{\phi=\phi_{\rm cl}}} + Z_n^{\rm 1PR}[J]\,, \tag{5.1}$$

where the contribution of 1PI diagrams to Z_n is obtained by evaluating Γ_n at the classical solution $\phi_{\rm cl} = \phi_{\rm cl}[J]$. In the following we will denote with a bar a functional which is evaluated at the classical solution: $\bar\Gamma := \Gamma_{|_{\phi=\phi_{\rm cl}}}$. We stress that the splitting of $Z_n[J]$ between 1PI and 1PR diagrams is ambiguous: either by a field redefinition, or by adding terms that vanish at the EoM (the two things are equivalent; see, e.g. [7]) one can change $\Gamma[\phi, J]$. On the other hand, if all counterterm Lagrangians $\mathcal{L}^{(k)}$ for all $k \le n$ have been specified, including terms that vanish at the EoM, then $\Gamma[\phi, J]$ is unambiguously defined.

The most important property of the effective action Γ in this context is that $Z_n^{\rm 1PR}$ can be written as tree diagrams with the Γ_k (with $k < n$) as vertices [1,2]. At the two- and three-loop level, e.g., we have

$$\begin{aligned} Z_2^{\rm 1PR} &= -\frac{1}{2}\bar\Gamma_1^i G_{il} \bar\Gamma_1^l\,, \\ Z_3^{\rm 1PR} &= \frac{1}{2}\bar\Gamma_1^i G_{ij} \bar\Gamma_1^{jk} G_{kl} \bar\Gamma_1^l - \bar\Gamma_2^i G_{ij} \bar\Gamma_1^j\,, \end{aligned} \tag{5.2}$$

where we have denoted functional derivatives with respect to ϕ with

$$\Gamma_n^{i_1 \dots i_k} := \frac{\delta^k \Gamma_n}{\delta\phi_{i_1} \dots \delta\phi_{i_k}}\,. \tag{5.3}$$

In general all vertices of the form $\Gamma_k^{j_1 \dots j_m}$ with $k < n$; $j_m \le n - k$ contribute to $Z_n^{\rm 1PR}$.

The condition that ensures that both $\bar\Gamma_n$ and $Z_n^{\rm 1PR}$ are separately finite can be established by induction. Suppose that for all $k < n$ $\bar\Gamma_k$ and $Z_k^{\rm 1PR}$ are separately finite. Since tree diagrams do not generate new divergences, $Z_n^{\rm 1PR}$ can be divergent only if some of the vertices are: in order to have $\bar Z_n^{\rm 1PR}$ finite we must impose that all the vertices $\bar\Gamma_k^{j_1 \dots j_m}$ with $k < n$ and $m \le n - k$ are finite. But all Γ_k for $k < n$ are finite at the EoM by assumption: possible divergences in their functional derivatives can be described by local terms that vanish at the EoM. These can be removed by tuning the coefficients of the counterterms of $\hbar$-order k that vanish at the EoM. The proof by induction is completed by observing that for $n = 0$ and 1 the generating functional does not admit divergent 1PR contributions: $\bar\Gamma_0$ and $\bar\Gamma_1$ are both finite (after renormalization). A detailed discussion of the case $n = 2$ can be found in [8].

In summary, in order to have 1PR and 1PI contributions to the generating functional Z separately finite, one must renormalize not only the $Z[J]$ but also the effective action $\Gamma[\phi, J]$ as a functional of ϕ. This can be done by exploiting the freedom to add counterterms that vanish at the EoM.

What we are interested in here, however, is not whether $Z_n^{\rm 1PR}$ is completely finite, but whether it plays a role in the RGE, and in particular in the HPRGE. According to the RGE the highest poles are fully determined by the $1/\varepsilon$ divergences of l-order one: in order not to have to consider 1PR graphs in the HPRGE, we must impose that $Z_n^{\rm 1PR}$ does not contain $1/\varepsilon$ divergences of l-order one. In view of the general statements made above about $Z_n^{\rm 1PR}$ this condition has to be transferred to the vertices Γ_k for all $k \le n - 1$. The part of Γ_k which is of l-order one can be projected out with the operator

$$P_{l=1}^k := \sum_{\alpha_i \beta_i = k-1} \prod_i \frac{1}{\beta_i!} \left[D^{(\alpha_i)}\right]^{\beta_i}\,, \tag{5.4}$$

where $[D^{(i)}]^1 = D^{(i)}$ and $\left[D^{(i)}\right]^{n+1} := \left(D^{(i)} - n\right)\left[D^{(i)}\right]^n$; cf. (2.14). The condition we have to impose at order n therefore reads

$$\left[P_{l=1}^k \Gamma_k\right]^{j_1 \dots j_m} = \text{finite} \tag{5.5}$$

for all $k < n$ and $m \le n - k$. At first sight, this may look like a severe complication of the implementation of the RGE. As we discuss in Appendix D, however, this is not the case: typically, for a given Γ_k only a finite number of functional derivatives may at all be divergent, such that whenever a new Γ_k is calculated and renormalized, one can impose that all its functional derivatives be finite (by an appropriate choice of the counterterm basis). If one renormalizes the theory in this manner the 1PR diagrams play no role in the renormalization procedure, and therefore in the RGE. An illustration of the concepts discussed here is provided in the following section for the case of renormalizable theories and in Appendix D in the general case.

6 RGE for renormalizable theories

The RGE we discussed so far were derived in a framework which is particularly convenient with non-renormalizable theories like CHPT. This setting, on the other hand, is completely general, and can be used also with renormalizable theories, as we want to show in this section. For the sake of simplicity we will discuss the case of a $O(N)$ ϕ^4 theory.

The $O(N)$ ϕ^4 theory is defined by the following classical Lagrangian:

$$\mathcal{L}^{(0)} = \frac{1}{2}\left(\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - M^2\vec{\phi}\cdot\vec{\phi}\right) - \frac{\lambda}{4}\left(\vec{\phi}\cdot\vec{\phi}\right)^2 - \vec{\phi}\cdot\vec{f}, \tag{6.1}$$

where $\vec{\phi}$ and $\vec{f}$ are N-component vectors, the latter of external fields. This theory is renormalizable: the divergences arising in loop calculations can be reabsorbed by a redefinition of the wave function renormalization Z_ϕ and the bare parameters M^2 and λ. Here, however, we want to discuss renormalization in a manner which is completely analogous to the case of a non-renormalizable theory. We will introduce a new Lagrangian at each order in $\hbar$:

$$\mathcal{L}^{\text{bare}} = \mathcal{L}^{(0)} + \hbar\mathcal{L}^{(1)} + \hbar^2\mathcal{L}^{(2)} + \dots, \tag{6.2}$$

with

$$\mathcal{L}^{(n)} = c_1^{(n)}\frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - c_2^{(n)}\frac{1}{2}M^2\vec{\phi}\cdot\vec{\phi} - c_3^{(n)}\frac{\lambda}{4}\left(\vec{\phi}\cdot\vec{\phi}\right)^2. \tag{6.3}$$

Notice that by using the equations of motion one can eliminate one of the four possible terms (6.1) such that a minimal basis of counterterms counts only three independent operators.

In order to renormalize the theory it is sufficient to define the bare couplings $c_i^{(n)}$ as

$$c_i^{(n)} = \left(\mu^{-\varepsilon}\lambda\right)^n\left[c_i^{(n)\,r} + \sum_{k=1}^{n} a_{k\,i}^{(n)}\varepsilon^{-k}\right], \tag{6.4}$$

where, for convenience we have factored out $\mu^{-\varepsilon n}\lambda^n$ in such a way that the renormalized couplings $c_i^{(n)\,r}$ and the coefficients $a_{k\,i}^{(n)}$ are dimensionless even in $d \neq 4$. With this choice the scaling with μ of the bare Lagrangians $\mathcal{L}^{(n)}$ is exactly as given in (2.5): the RGE that follow from there must therefore be valid also in this case. The coefficients $a_{k\,i}^{(n)}$ are analogous to those defined in (2.8), and can further be expanded in the l-order: $a_{k\,i}^{(n)} = \sum_{l=k}^{n} a_{lk\,i}^{(n)}$. Each new coupling constant $c_i^{(n)\,r}$ has its own beta function, defined as

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}c_i^{(n)\,r}(\mu) = \beta_i^{(n)} + \varepsilon n c_i^{(n)}, \tag{6.5}$$

and which the RGE relate to:

$$\beta_i^{(n)} = \sum_{l=1}^{n} l a_{l1\,i}^{(n)}. \tag{6.6}$$

A full two-loop calculation of the generating functional in this theory is relatively easy – the results can be found, e.g., in [13] and read

$$a_{1\,1}^{(1)} = 0, \quad a_{1\,2}^{(1)} = \frac{2}{P}(N+2), \quad a_{1\,3}^{(1)} = \frac{2}{P}(N+8), \tag{6.7}$$

for the one-loop divergences and

$$\begin{aligned}
a_{1\,1}^{(2)} &= -\frac{1}{P^2}(N+2)\,,\\
a_{2\,1}^{(2)} &= 0\,,\\
a_{1\,2}^{(2)} &= -\frac{2}{P^2}(N+2)\left[3 - P(c_2^{(1)\,r} + c_3^{(1)\,r})\right],\\
a_{2\,2}^{(2)} &= \frac{4}{P^2}(N+2)(N+5)\,,\\
a_{1\,3}^{(2)} &= -\frac{4}{P^2}\left[22 + 5N - P(N+8)c_3^{(1)\,r}\right],\\
a_{2\,3}^{(2)} &= \frac{4}{P^2}(N+8)^2\,,
\end{aligned} \tag{6.8}$$

at two loops, with $P = 16\pi^2$. The results of the two-loop divergences allow us to test the RGE for $n = 2$ which are written explicitly in (3.2):

$$\begin{aligned}
2a_{2\,1}^{(2)} &= \nabla a_{1\,1}^{(2)} = \sum_{i=1}^{3} a_{1\,i}^{(1)}\frac{\partial}{\partial c_i^{(1)\,r}}a_{1\,1}^{(2)} = 0\,,\\
2a_{2\,2}^{(2)} &= \nabla a_{1\,2}^{(2)} = \sum_{i=1}^{3} a_{1\,i}^{(1)}\frac{\partial}{\partial c_i^{(1)\,r}}a_{1\,2}^{(2)}\\
&= \frac{4}{P^2}(N+2)(2N+10)\,,\\
2a_{2\,3}^{(2)} &= \nabla a_{1\,3}^{(2)} = \sum_{i=1}^{3} a_{1\,i}^{(1)}\frac{\partial}{\partial c_i^{(1)\,r}}a_{1\,3}^{(2)} = \frac{8}{P^2}(N+8)^2\,,
\end{aligned} \tag{6.9}$$

which perfectly agree with the results of the direct calculation (6.8). We can now extend and solve the RGE for the highest poles to all orders. What makes such a solution possible in a renormalizable theory is the fact that the new vertices coming from the $\mathcal{L}^{(n)}$ Lagrangian (and that have to be inserted in the relevant one-loop diagrams) are identical to those appearing in the classical Lagrangian (6.1).

A divergent one-loop contribution to the renormalization of $c_3^{(n)}$ can have at most two vertices, i.e. must be proportional to $c_3^{(n-k)\,r}c_3^{(k-1)\,r}$, with $k = 1, \dots, n$. The coefficient of all these terms is identical, and can be read off from (6.7):

$$a_{11\,3}^{(n)} = \frac{2}{P}(N+8)\sum_{k=1}^{n} c_3^{(n-k)\,r}c_3^{(k-1)\,r}. \tag{6.10}$$

The equation to all orders (3.11) can now be easily solved by induction. We have seen that

$$a_{1\,3}^{(1)} = p, \qquad a_{2\,3}^{(2)} = p^2, \qquad p := \frac{2(N+8)}{P}, \tag{6.11}$$

and it is easy to prove that if $a_{n-1\,3}^{(n-1)} = p^{n-1}$, then (6.10) and (3.11) imply that $a_{n\,3}^{(n)} = p^n$.

The solution of the RGE for the mass term is only slightly more complicated but can also be solved to all orders. The divergent part of all one-loop graphs at $\hbar$-order n can also in this case depend on at most two counterterms, $c_3^{(k)\,r}c_2^{(n-1-k)\,r}$, for all $0 \leq k \leq n-1$. By exploiting again the fact that the structure of the counterterms is identical to that of the classical Lagrangian we can read off from (6.7) the coefficient of all the divergent terms:

$$a_{11\,2}^{(n)} = \frac{q}{n}\sum_{k=0}^{n-1} c_3^{(k)\,r}c_2^{(n-1-k)\,r}, \qquad q := \frac{2}{P}(N+2). \tag{6.12}$$

At one and two loops we had

$$a^{(1)}_{1\,2} = q, \qquad a^{(2)}_{2\,2} = \frac{1}{2}q(q+p)\,, \tag{6.13}$$

and by using (6.12) and (3.11) we can easily prove by induction that, if

$$a^{(n-1)}_{n-1\,2} = \frac{1}{(n-1)!}\prod_{k=0}^{n-2}(q+kp)\,, \tag{6.14}$$

then

$$a^{(n)}_{n\,2} = \frac{1}{n!}\prod_{k=0}^{n-1}(q+kp)\,. \tag{6.15}$$

6.1 EoM counterterms

The example of the $O(N)$ ϕ^4 Lagrangian is useful also to illustrate what is the role of EoM counterterms. For the $\mathcal{L}^{(n)}$ Lagrangians we chose above the set of operators

$$\begin{aligned}\mathcal{O}_1 &= \frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi}, \qquad \mathcal{O}_2 = -\frac{1}{2}M^2\vec{\phi}\cdot\vec{\phi},\\ \mathcal{O}_3 &= -\frac{\lambda}{4}\left(\vec{\phi}\cdot\vec{\phi}\right)^2,\end{aligned} \tag{6.16}$$

but we could in principle choose a different set at each order n, and replace one of the operators in (6.16) with

$$\mathcal{O}_4 = -\vec{f}\cdot\vec{\phi}\,. \tag{6.17}$$

The general discussion given in Sect. 5 shows that if one wants to deal only with 1PI diagrams in the RGE, one should select the counterterm Lagrangian requiring that even functional derivatives with respect to ϕ of the effective action Γ_n be finite. In the present case, at one loop we have

$$\begin{aligned}\Gamma_1 &= S_1 + \frac{1}{2}\mathrm{Tr}\log S_0^{ij}\,,\\ \Gamma_1^k &= S_1^k + \frac{1}{2}S_0^{ijk}G_{ij}\,.\end{aligned} \tag{6.18}$$

The evaluation of the loop part of Γ_1^k gives

$$\begin{aligned}\frac{1}{2}S_0^{ijk}G_{jk} = -\frac{1}{\varepsilon}\frac{2}{P}\left\{\phi_i\left[M^2(N+2)+\lambda\vec{\phi}\cdot\vec{\phi}(N+8)\right]\right\}\\ +\text{finite terms}\,,\end{aligned} \tag{6.19}$$

while the functional derivative of the contribution to the action of the four operators entering at each order reads

$$\begin{aligned}\frac{\delta\mathcal{O}_1}{\delta\phi_i} &= -\partial^2\phi_i\,, & \frac{\delta\mathcal{O}_2}{\delta\phi_i} &= -M^2\phi_i\,,\\ \frac{\delta\mathcal{O}_3}{\delta\phi_i} &= -\lambda\phi_i\left(\vec{\phi}\cdot\vec{\phi}\right), & \frac{\delta\mathcal{O}_4}{\delta\phi_i} &= -f_i\,,\end{aligned} \tag{6.20}$$

which shows that the divergent part in $S_0^{ijk}G_{jk}/2$ can be cancelled by $\bar{S}_1^i$ only if both $\mathcal{O}_2$ and $\mathcal{O}_3$ are included in the list of operators: at one loop either $\mathcal{O}_1$ or $\mathcal{O}_4$ can be eliminated. In fact, since the structure of the Lagrangian is always the same, this criterion extends also to higher orders.

In this example it is also easy to verify (and we leave this to the reader) that in case one choses a different basis for the counterterm Lagrangian, e.g. $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_4$, then 1PR diagrams do contribute to local divergences, and that if one takes them also into account then the RGE still hold as before. Fixing the basis in such a way that only 1PI graphs contribute to the divergences is only a matter of convenience and does not change the form of the RGE.

6.2 Comparison to the standard treatment of $\lambda\phi^4$

The way we discussed renormalization for this example of a renormalizable theory is not standard, as it is designed to parallel the renormalization procedure in CHPT (or any other non-renormalizable theory). In this section we will clarify the connection between our treatment and the standard one.

The multiplicity of coupling constants we introduced, the $c_i^{(n)}$, are not separately observable. Indeed, the physical mass and coupling constant that will appear in any observable will have the form

$$\begin{aligned}M^2_{\rm ph} &= M^2\left(1+\sum_{n=1}^{\infty}a^{(n)}_{n\,2}\ell^n+\ldots+\sum_{n=1}^{\infty}\lambda^n c_2^{(n)\,r}(\mu)\right)\\ &\simeq M^2\left(1-p\ell\right)^{-q/p}\,,\\ \lambda_{\rm ph} &= \lambda\left(1+\sum_{n=1}^{\infty}a^{(n)}_{n\,3}\ell^n+\ldots+\sum_{n=1}^{\infty}\lambda^n c_3^{(n)\,r}(\mu)\right)\\ &\simeq \frac{\lambda}{1-p\ell}\,,\end{aligned} \tag{6.21}$$

where $\ell = \lambda\log\mu$, and where the ellipses denote terms with subdominant powers of logs, and where the last expression is accurate only up to the leading logs. Notice that the expression between brackets after the first equality sign in (6.21) is μ-independent: λ and M are μ-independent by definition, and the coefficients of each power of λ are separately μ-independent as implied by the β-functions of the $c_i^{(n)\,r}$ couplings.

Since λ and $c_3^{(i)\,r}(\mu)$ always appear in this combination, and are not separately observable, it is useful to lump them together into one single quantity, the part in $\lambda_{\rm ph}$ which does not contain logs:

$$\lambda_{\rm R}(\mu) := \lambda\left(1+\sum_{i=1}^{\infty}\lambda^i c_3^{(i)\,r}(\mu)\right), \tag{6.22}$$

and which is nothing but the standardly defined renormalized coupling constant. Any quantity can now be expressed in terms of $\lambda_{\rm R}(\mu)$ (and the similarly defined renormalized mass), rather than λ, by inverting (6.22).

It is interesting to derive how $\lambda_{\rm R}(\mu)$ depends on μ from (6.22):

$$\beta_\lambda := \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\lambda_{\rm R}(\mu) = \nabla\lambda_{\rm R}(\mu) = \lambda\sum_{i=1}^{\infty}\lambda^i\beta_3^{(i)}\,, \qquad (6.23)$$

and if we re-express λ on the RHS in terms of $\lambda_{\rm R}(\mu)$ we finally get

$$\beta_\lambda = \lambda_{\rm R}(\mu)\sum_{i=1}^{\infty}\lambda_{\rm R}^i(\mu)\bar\beta_3^{(i)}\,, \qquad (6.24)$$

where

$$\bar\beta_3^{(i)} = \left[\beta_3^{(i)}\right]\Big|_{c_i^{(n)}=0} \qquad (6.25)$$

is the part of the beta functions which does not depend on any of the constants. (Which can also be identified with the help of the loop number: $\beta_3^{(i)} = \sum_l \beta_{l\,3}^{(i)}$; $\bar\beta_3^{(i)} = \beta_{i\,3}^{(i)}$.)

7 RGE for theories with renormalizable and non-renormalizable interactions

In the previous section we have shown how the RGE derived here apply to the case of a renormalizable field theory. Our whole framework, on the other hand, has been developed in order to treat the case of a non-renormalizable field theory like CHPT, where the only renormalizable part of the Lagrangian corresponds to a free field theory (in the chiral limit). Between these two extreme cases there is an intermediate one, which is actually used quite often in phenomenology: the case of an interacting renormalizable theory to which one adds non-renormalizable interactions. The latter are usually suppressed by powers of a certain energy scale. For example the search for new physics beyond the standard model is often performed in such a framework – translating experimental measurements into bounds on the strength of the non-renormalizable interactions. In the present section we will discuss the application of our RGE to such cases. The role of EoM terms in such a framework has been discussed in [16]. As in the previous section we choose to work with a simple example and illustrate in the clearest possible setting the use of the RGE. We take as renormalizable part of the Lagrangian the $O(N)$ ϕ^4 theory discussed above and add a ϕ^6 term to it:

$$\begin{aligned}\mathcal{L}^{(0)} = {}& \frac{1}{2}\left(\partial_\mu\vec\phi\cdot\partial^\mu\vec\phi - M^2\vec\phi\cdot\vec\phi\right) - \frac{\lambda}{4}\left(\vec\phi\cdot\vec\phi\right)^2 \\ & -\frac{g}{6\Lambda^2}\left(\vec\phi\cdot\vec\phi\right)^3 - \vec\phi\cdot\vec f\,,\end{aligned} \qquad (7.1)$$

where g is a dimensionless coupling constant and Λ an energy scale. Such a Lagrangian is non-renormalizable, i.e. more and more counterterms will be required in order to make loop calculations finite. We choose, however, to neglect all the effects which are suppressed by more powers of Λ than the ϕ^6 interaction. With this choice, even at higher orders we will consider only the four operators introduced in (7.1):

$$\begin{aligned}\mathcal{L}^{(n)} = {}& c_1^{(n)}\frac{1}{2}\partial_\mu\vec\phi\cdot\partial^\mu\vec\phi - c_2^{(n)}\frac{1}{2}M^2\vec\phi\cdot\vec\phi \\ & -c_3^{(n)}\frac{\lambda}{4}\left(\vec\phi\cdot\vec\phi\right)^2 - c_4^{(n)}\frac{g}{6\Lambda^2}\left(\vec\phi\cdot\vec\phi\right)^3\,.\end{aligned} \qquad (7.2)$$

If we define the bare couplings as in (6.4) and renormalize the theory at the one-loop level we find the following non-zero coefficients:

$$\begin{aligned}a_{1\,2}^{(1)} &= \frac{2}{P}(N+2),\\ a_{1\,3}^{(1)} &= \frac{2}{P}\left[(N+8)+2\eta(N+4)\right],\\ a_{1\,4}^{(1)} &= \frac{6}{P}(N+14)\,,\end{aligned} \qquad (7.3)$$

where

$$\eta := \frac{gM^2}{\lambda^2\Lambda^2}\,. \qquad (7.4)$$

The RGE allow us to move to higher loops: at the two-loop level, e.g. we find

$$\begin{aligned}a_{2\,2}^{(2)} &= \frac{1}{P}(N+2)\left[a_{1\,2}^{(1)} + a_{1\,3}^{(1)}\right],\\ \tilde a_{2\,3}^{(2)} &= \frac{2}{P}\left[(N+8)a_{1\,3}^{(1)} + \eta(N+4)\left(a_{1\,2}^{(1)} + a_{1\,4}^{(1)}\right)\right],\\ a_{2\,4}^{(2)} &= \frac{3}{P}(N+14)\left(a_{1\,3}^{(1)} + a_{1\,4}^{(1)}\right),\end{aligned} \qquad (7.5)$$

where, for convenience, we have introduced the symbol

$$\tilde a_{n\,3}^{(n)} = a_{n\,3}^{(n)} + \eta\hat a_{n\,3}^{(n)}\,, \qquad (7.6)$$

which explicitly shows that for $\eta = 0$ we obtain exactly the same coefficients $a_{n\,3}^{(n)}$ as in the renormalizable case. If we apply the RGE to higher orders following the same reasoning used in the previous section we get the general results

$$\begin{aligned}\hat a_{n\,3}^{(n)} &= \frac{1}{n}\frac{4}{P}(N+4)\sum_{k=1}^{n} a_{n-k\,2}^{(n-k)}a_{k-1\,4}^{(k-1)},\\ a_{n\,4}^{(n)} &= \frac{1}{n}\frac{6}{P}(N+14)\sum_{k=1}^{n} a_{n-k\,3}^{(n-k)}a_{k-1\,4}^{(k-1)}\,,\end{aligned} \qquad (7.7)$$

where we have considered only the new parts with respect to the purely renormalizable case. These recursion relations can be solved also in this case and give

$$\hat a_{n\,3}^{(n)} = \frac{r}{n!}\prod_{k=0}^{n-1}(t+kp)\,, \qquad a_{n\,4}^{(n)} = \frac{1}{n!}\prod_{k=0}^{n-1}(s+kp)\,, \qquad (7.8)$$

where p has been introduced in the previous section and

$$\begin{aligned}r &:= \frac{4(N+4)}{P}\,, \qquad s := \frac{6(N+14)}{P}\,,\\ t &:= q+s = \frac{8(N+11)}{P}\,.\end{aligned} \qquad (7.9)$$

In our formulation of the renormalization procedure an infinite number of finite counterterms appear, which are not individually observable, as already discussed for the ϕ^4 theory. Only the sum over the series of counterterms

and the accompanying logs are observable quantities. For the coupling constant of the new ϕ^6 interaction we find

$$g_{\rm ph} = g\left(1+\sum_{n=1}^{\infty} a_{n\,4}^{(n)}\ell^n + \ldots + \sum_{n=1}^{\infty}\lambda^n c_4^{(n)\,r}(\mu)\right) + O\left(\frac{g^2M^2}{\Lambda^2}\right) \simeq g(1-\ell p)^{-s/p} + O\left(\frac{g^2M^2}{\Lambda^2}\right), \quad (7.10)$$

which shows that even to first order in g the observable coupling constant gets renormalized in a non-trivial way by the renormalizable part of the interaction, and that the corresponding series of leading logs can be resummed. The results obtained here are in agreement with what one would obtain by considering the standard treatment of the scaling behavior of operators – the solution of our general RGE in this particular case has shown that the series of the leading logs is determined by one single parameter, s, which is nothing but the anomalous dimension (modulo normalization factors) of the ϕ^6 operator.

As far as the renormalizable ϕ^4 interaction is concerned, we have seen that the ϕ^6 term renormalizes it at every order in the loop expansion. The renormalization is proportional to gM^2/Λ^2 and requires the introduction of a specific counterterm: the one for the ϕ^4 term now must have the form

$$\tilde{c}_3^{(n)} = c_3^{(n)} + \frac{gM^2}{\Lambda^2}\hat{c}_3^{(n)}, \quad (7.11)$$

where both $c_3^{(n)}$ and $\hat{c}_3^{(n)}$ have to be split into infinite and finite, scale-dependent parts. The finite, observable coupling constant now becomes

$$\lambda_{\rm ph} = \lambda\left(1+\sum_{n=1}^{\infty} a_{n\,3}^{(n)}\ell^n + \ldots + \sum_{n=1}^{\infty}\lambda^n c_3^{(n)\,r}(\mu)\right) + r\frac{gM^2}{\Lambda^2}\left(1+\sum_{n=1}^{\infty}\hat{a}_{n\,3}^{(n)}\ell^n + \ldots + + \sum_{n=1}^{\infty}\lambda^n \hat{c}_3^{(n)\,r}(\mu)\right). \quad (7.12)$$

At first sight one may get the impression that, if we were now to resum the series of the leading logs in $\lambda_{\rm ph}$, and define the corresponding $\lambda_{\rm R}(\mu)$, this would scale differently from the renormalizable case. However we first notice that the correction proportional to gM^2/Λ^2 in (7.12) is scale independent. Indeed one can easily check that the series of the leading logs in the second term on the right-hand side of (7.12) can be fully reabsorbed by substituting $gM^2/\Lambda^2 \to g_{\rm ph}M_{\rm ph}^2/\Lambda^2$. The leading log approximation for $\lambda_{\rm ph}$ therefore reads

$$\lambda_{\rm ph} \simeq \frac{\lambda}{1-p\ell} + r\frac{g_{\rm ph}M_{\rm ph}^2}{\Lambda^2} + O(g^2) \simeq \frac{\tilde{\lambda}}{1-p\ell}, \quad (7.13)$$

with $\tilde{\lambda} = \lambda + r g_{\rm ph}M_{\rm ph}^2/\Lambda^2$. The conclusion is that even in the presence of a ϕ^6 interaction the scaling behavior of the ϕ^4 coupling constant does not change, provided one uses the same renormalization condition for λ.

8 Conclusions

In this paper we have studied the RGE for a generic non-renormalizable QFT. In the formulation of the problem we have adopted a notation suited to the case of CHPT, but have not used any of its specific properties in the derivation of the equations: the RGE that we derived are completely general.

We have worked out explicitly the structure of the leading divergences to all orders and found that they can be recursively expressed in terms of divergences of one-loop diagrams only. This result is an extension to all orders of the result obtained by Weinberg at the two-loop level [3]. Like in that case, however, where in order to obtain the leading two-loop divergence one had to perform a new and non-trivial one-loop calculation [6, 8], the extension to higher loops also requires at each step a new one-loop calculation. In the case of CHPT, e.g., such one-loop calculations are, although straightforward in principle, long and tedious in practice. As we do not know a way to perform all these calculations in one go, and solve explicitly the recursive procedure, we are not able to provide a method to make resummations of series of leading chiral logs.

A technical problem which occurs in the practical use of the RGE concerns the role of 1PR diagrams. This in turn is related to the freedom one has in choosing a basis for the counterterms at each order in the perturbative expansion, and to the fact that different bases may be related by counterterms that vanish at the solution of the equations of motion. As we have shown one can use this freedom to choose the basis at each order in such a way that in the RGE only 1PI one-loop diagrams have to be considered. Alternatively, if one wants to use an arbitrary basis then the RGE provide the right answer for the leading divergences only if one takes into account also the local divergences generated by 1PR one-loop graphs.

We have analyzed also the RGE for the subleading divergences, but there even a fully explicit recursive relation is too complicated to write down. We have discussed explicitly the equations at the two- and three-loop level. A discussion of how one can derive the all-order formula can be found in Appendix B.

If one formulates the renormalization procedure for a renormalizable QFT by introducing at each loop order (or order in $\hbar$) a new bare Lagrangian which is independently scale invariant, the RGE which we have derived here apply equally well to this case. We have shown in the explicit example of a ϕ^4 theory that one can solve explicitly the recursion relations for the leading divergences and obtain results which are in full agreement with those obtained in the usual formulation of renormalizable QFT. We could calculate explicitly the series of the leading divergences even after adding a ϕ^6 interaction to the ϕ^4 theory: in this case the results provided a calculation of the anomalous dimension of the ϕ^6 operator. These explicit examples illustrate neatly why the complicated structure of the RGE that we have derived becomes manageable for the case of a renormalizable theory: the structure of the counterterm Lagrangian is the same to all orders in $\hbar$, and this makes the solution of the recursion relations possible.

This is unfortunately not the case for non-renormalizable theories of the CHPT kind, where a resummation of the leading divergences does not seem to be feasible. In the past, applications of the RGE in CHPT have concerned the calculations of double chiral logs for various quantities. We plan to extend these calculations to other quantities of interest, namely weak non-leptonic decays, where these double chiral logs will provide interesting information about the next-to-next-to-leading corrections. It will also be very interesting to explore the practical feasibility of calculations of triple chiral logs for some simple quantities, e.g. M_π, and see how far one can push the calculation of the series of the leading logs [17].

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A Dimensional analysis

In deriving the RGE we have used as starting point the scaling with μ of the Lagrangians of $\hbar$-order n given in (2.5), and have justified this choice with the claim that in CHPT this is the correct one. We explain this here. The leading order CHPT Lagrangian (for simplicity we work here in the chiral limit) reads [4]

$$\mathcal{L}^{(0)} = \frac{F^2}{4}\langle \partial_\mu U^\dagger \partial^\mu U\rangle\,, \tag{A.1}$$

with $U = \exp \mathrm{i}\phi/F$ a dimensionless function of ϕ, which implies $[\phi] = [F]$. The dimension of $\mathcal{L}^{(0)}$ (which is μ-independent by definition) is d, which implies $d = 2[\partial_\mu] + 2[F]$. This leads to $[\phi] = [F] = \frac{d-2}{2}$.

At higher orders the Lagrangians $\mathcal{L}^{(n)}$ will contain $2(n+1)$ powers of derivatives which at $d = 4$ must be compensated by $2(n+1) - 4 = 2(n-1)$ inverse powers of a physical energy scale. The only available one in this framework is F. The correct dimensions of $\mathcal{L}^{(n)}$ for $d \neq 4$ can be restored by the appropriate powers of the arbitrary scale μ. The Lagrangian $\mathcal{L}^{(n)}$ must therefore scale with a factor $\mu^{-\varepsilon n}$.

B Proof of the highest-pole equation to all orders

B.1 Notation

In the RGE products of the ∇_l appear everywhere, and in order to fully exploit the information contained in the RGE it is necessary to express them explicitly. In this appendix we show how to do this. We first introduce some convenient notation, and denote a product of k ∇_{l_i}s simply by the (ordered) list of l_is within square brackets:

$$\nabla_{l_1}\nabla_{l_2}\ldots\nabla_{l_k} =: [l_1 l_2 \ldots l_k]\,. \tag{B.1}$$

We remind the reader that the subscripts l_i stand for the l-order of the β-functions appearing inside the ∇. Such a product contains many terms because each ∇_{l_i} can act on all other ∇_{l_j} on its right-hand side. In order to handle these many terms conveniently it is necessary to find a sufficiently compact notation. We illustrate this by considering first the simple case of a product of two ∇s. This has two terms, one where the derivative in the first ∇ acts on the β in the second ∇, and one where both derivatives are free to act on whatever is on their right-hand side:

$$\begin{aligned}\nabla_l\nabla_k &= \left(\sum_n \vec{\beta}_l^{(n)}\cdot\vec{\partial}^{(n)}\right)\left(\sum_m \vec{\beta}_k^{(m)}\cdot\vec{\partial}^{(m)}\right) \\ &= \sum_{n,m}\left[\beta_{li}^{(n)}\frac{\partial\beta_{kj}^{(m)}}{\partial c_i^{(n)}}\frac{\partial}{\partial c_j^{(m)}} + \beta_{li}^{(n)}\beta_{kj}^{(m)}\frac{\partial}{\partial c_i^{(n)}}\frac{\partial}{\partial c_j^{(m)}}\right],\end{aligned} \tag{B.2}$$

where we have used the summation convention for repeated indices. We denote these two terms by

$$[lk] = (l,k) + (l)(k)\,. \tag{B.3}$$

To each bracket corresponds one free derivative, and the brackets commute by definition. Consider now a product of three ∇s, and construct it by multiplying from the left the product of two ∇s with another ∇. The latter can either act on the other two ∇s or remain free. In the notation just introduced this can be rephrased as follows: if we add a new index from the left in the left-hand side of (B.3) this can enter from the left on each of the brackets on the right-hand side of (B.3), or stay alone in its own bracket:

$$\begin{aligned}[jlk] = {}&([jl],k) + (k)(j,l) + (l)(j,k) \\ &+(j)(l,k) + (j)(l)(k)\,.\end{aligned} \tag{B.4}$$

Using (B.3) the first term could still be rewritten as

$$([jl],k) = ((j,l),k) + ((j)(l),k)\,. \tag{B.5}$$

In general a bracket can have at most two arguments: the first one (if present) is a ∇, or a product thereof, whose free derivatives are all acting on the ∇ identified by the second argument. The derivative corresponding to the latter ∇ remains free.

Consider now a generic product of k ∇s. It is easy to convince oneself that one can write this as

$$[l_1\ldots l_k] = \sum_{\substack{\text{all}\\ \text{splittings}}}\prod_{\substack{\text{all}\\ \text{subsets}}}([l_{i_1}\ldots l_{i_{j-1}}], l_j)\,, \tag{B.6}$$

and that a recursive use of the latter will generate all the terms in the product. Note that the sum has to run over all possible splittings of the set of k numbers into subsets, and that the order of the l_i inside the subsets has to be the same as in the original set.

Until now all the indices appearing in this notation referred to the l-order, but it is useful also to introduce an index related to the v-order. A given ∇_l contains derivatives with respect to coupling constants of any v-order larger than l:

$$\nabla_l = \sum_{n=l}^{\infty} \vec{\beta}_l^{(n)} \cdot \vec{\partial}^{(n)} \,. \tag{B.7}$$

A ∇ (or a product thereof), however, always acts on objects (the $\mathcal{A}_l^{(n)k}$) which are polynomials in the coupling constants $c_i^{(n)}$, and which have a maximum v-order, such that only a finite number of terms in the infinite sum (B.7) will play a role. The v-order of a monomial in the $c_i^{(n)}$ is defined as

$$N_v \left[\prod_i \left(c_{j_i}^{(n_i)} \right)^{k_i} \right] = \sum_i n_i k_i \,. \tag{B.8}$$

Analogously, a product of derivatives will reduce the v-order of the object it acts on by the amount

$$\Delta_v \left[\prod_i \left(\partial_{j_i}^{(n_i)} \right)^{k_i} \right] = \sum_i n_i k_i \,. \tag{B.9}$$

A product of derivatives acting on a monomial gives zero if the v-order of the latter is lower than the Δ_v of the derivatives. For a polynomial in the $c_i^{(n)}$ it is important to identify its maximum v-order: for a $\mathcal{A}_{lk}^{(n)}$ this is equal to $n-l$ (which is also its minimum v-order). The first RGE relates $\mathcal{B}_l^{(n)}$ to $\mathcal{A}_{l1}^{(n)}$: the v-order of $\mathcal{B}_l^{(n)}$ is therefore also equal to $n-l$.

Equation (B.7) shows that the minimal Δ_v of ∇_l is l. If we consider products of ∇s, their minimal Δ_v is

$$\min\left(\Delta_v[l_1 \dots l_k]\right) = \sum_i l_i \,. \tag{B.10}$$

It is convenient to write the brackets as sums of terms with a definite v-order:

$$([l_1 \dots l_k], j) = \sum_{n=l_1+\ldots+l_k+j}^{\infty} ([l_1 \dots l_k], j)_n \,, \tag{B.11}$$

where we have made explicit the fact that the minimal Δ_v of a bracket is equal to the sum of all the indices inside the bracket (B.10). Moreover, according to (B.9), the Δ_v of a product of brackets is equal to the sum of their Δ_vs:

$$\Delta_v \Big[([\dots], l_1)_{n_1} ([\dots], l_2)_{n_2} \dots ([\dots], l_j)_{n_j} \Big] = \sum_{i=1}^{j} n_i \,. \tag{B.12}$$

B.2 Highest pole equation

The equation for the highest pole reads

$$n! \mathcal{A}_n^{(n)n} = \nabla_1^{n-1} \beta_1^{(n)} \,. \tag{B.13}$$

Note that this is actually a set of equations for each of the components of $\mathcal{A}_n^{(n)n}$ and $\beta_1^{(n)}$:

$$\mathcal{A}_{nn}^{(n)} = \vec{a}_{nn}^{(n)} \cdot \vec{\mathcal{O}}^{(n)} \,, \quad \mathcal{B}_1^{(n)} = \vec{\beta}_1^{(n)} \cdot \vec{\mathcal{O}}^{(n)} \,, \tag{B.14}$$

and that the operators $\mathcal{O}_i^{(n)}$ just play the role of a basis of vectors and only allow us to write the equation more compactly. In fact the content of the equations remains exactly the same if we substitute $\mathcal{O}_i^{(n)} \to \partial_i^{(n)}$. If we do that, (B.13) gets rewritten as

$$n! d_n = \big([\underbrace{11 \dots 1}_{n-1}]; 1 \big)_n = \big([1^{n-1}], 1 \big)_n \,, \tag{B.15}$$

where we have directly used the notation with the brackets, and where we have introduced the symbol $[1^{n-1}]$ for the product of $n-1$ ∇_1.

In order to express fully explicitly the highest poles $\mathcal{A}_{nn}^{(n)}$, or equivalently the d_n, we now have to use (B.6) and split the $n-1$ ∇_1 in all possible subsets:

$$[1^{n-1}]_{n-1} = \sum_{\{n_i m_i = n-1\}} (n-1)! \prod_i c_i \Big[\big([1^{n_i - 1}], 1 \big)_{n_i} \Big]^{m_i} \,, \tag{B.16}$$

where c_i is a combinatorial factor that we will discuss below. Note that we have used explicitly the fact that the total Δ_v has to be equal to n: this implies that all brackets can contribute only with their minimal Δ_v – according to (B.15) their contribution is equal to $n_i! d_{n_i}$.

Finally we have to discuss the factors c_i, which count how many times a term is generated in the expansion of the product of $n-1$ ∇_1s. We do this in the following steps.
(1) We first permute the n factors ∇_1 in all possible ways, and get a factor $(n-1)!$ (already written explicitly in (B.16)). We can now just count the different splittings of $n-1$ into smaller integers, and not consider the ordering. This however generates an overcounting, which is compensated for in the next two steps.
(2) The ordering of the ∇_1 in each subset has to be like the original ordering. To compensate for this overcounting we must include a factor $1/(n_i!)^{m_i}$ in c_i.
(3) The m_i copies of the same subset are not distinguishable: c_i must also contain a factor $1/m_i!$. In total we get

$$c_i = \frac{1}{(n_i!)^{m_i} m_i!} \,, \tag{B.17}$$

and finally

$$n! d_n = (S_{n-1}, 1)_n \,, \tag{B.18}$$

where

$$S_n := [1^n]_n = n! \sum_{\{n_i m_i = n\}} \prod_i \frac{1}{m_i!} d_{n_i}^{m_i} \,, \tag{B.19}$$

which is the result we were after.

C Beyond the highest-pole equation

We will now consider the divergences $\mathcal{A}_{lk}^{(n)}$ with $k < n$. The starting point is the RGE (2.21) which we rewrite

here for convenience:

$$l\mathcal{A}^{(n)}_{lk} = \sum_{l'=1}^{l-k+1} \nabla_{l'} \mathcal{A}^{(n)}_{l-l'\,k-1} \qquad l = k, \dots n, \;\; k = 2, \dots, n\,.$$

The equation relates $\mathcal{A}^{(n)}_{lk}$ to $\mathcal{A}^{(n)}_{l'\,k-1}$, but if we apply it recursively we end up relating it to $\mathcal{A}^{(n)}_{l'\,1} = 1/l'\mathcal{B}^{(n)}_{l'}$, in the following manner:

$$\mathcal{A}^{(n)}_{lk} = \sum_{\underline{l}} c_{\underline{l}} \nabla_{l_1} \cdot \ldots \cdot \nabla_{l_{k-1}} \mathcal{B}^{(n)}_{l_k}\,; \quad c_{\underline{l}} = \left(\prod_{j=1}^{k} \sum_{m=j}^{k} l_m \right)^{-1}, \tag{C.1}$$

where the sum runs over all possible ordered k-ple $\underline{l} = (l_1, \dots, l_k)$ with the property $\sum_{i=1}^{k} l_i = l$. Using the notation introduced in the previous section we can rewrite this as

$$d^{(n)}_{lk} = \sum_{\underline{l}} c_{\underline{l}} ([l_1 \dots l_{k-1}], l_k)_n\,, \tag{C.2}$$

where

$$\mathcal{A}^{(n)}_{lk} = \vec{a}^{(n)}_{lk} \cdot \vec{\mathcal{O}}^{(n)} \;\Rightarrow\; d^{(n)}_{lk} = \vec{a}^{(n)}_{lk} \cdot \vec{\partial}^{(n)}\,. \tag{C.3}$$

If we want to make this equation more explicit we have to expand the bracket $[l_1 \dots l_{k-1}]$ according to (B.6) and relate the various brackets to ds with lower $\hbar$-order. The scheme is recursive and allows one to go to as high an $\hbar$-order as one wants, but having a fully explicit formula to all orders like the one we had for the highest pole looks very difficult. In order to illustrate what kind of difficulties one faces beyond the leading poles we will now discuss the case of the subleading poles.

In the case of the subleading poles ($k = n-1$) we have to deal with two different l-order. Equation (C.2) can be written down as follows for this case:

$$\begin{aligned} (n-1)!\, d^{(n)}_{n-1\,n-1} &= \left([1^{n-2}], 1\right)_n \\ n!\, d^{(n)}_{n\,n-1} &= \left([1^{n-2}], 2\right)_n \\ &\quad + \sum_{j=0}^{n-3} (n-1-j) \left([1^j 2 1^{n-3-j}], 1\right)_n\,, \end{aligned} \tag{C.4}$$

where

$$[1^j 2 1^k] := [\underbrace{1 \dots 1}_{j} 2 \underbrace{1 \dots 1}_{k}]\,. \tag{C.5}$$

We first consider the first equation in (C.4): the right-hand side has $\Delta_v = n$ and therefore can be split into two terms:

$$\left([1^{n-2}], 1\right)_n = \left([1^{n-2}]_{n-2}, 1\right)_n + \left([1^{n-2}]_{n-1}, 1\right)_n\,. \tag{C.6}$$

The first term can be written down explicitly: $[1^{n-2}]_{n-2}$ is just S_{n-2}, whose expression is given in (B.19). As for the second term, it has the same structure as S_{n-2}, but one order of Δ_v higher. It is useful to introduce a new symbol for $[1^n]_{n+1}$:

$$S^1_n := [1^n]_{n+1} = n! \sum_{n_0=1}^{n} \left[\frac{1}{(n-n_0)!} d^{(n_0+1)}_{n_0\,n_0} S_{n-n_0} \right], \tag{C.7}$$

where the latter expression can be obtained with the following reasoning: one starts by expanding $[n]$ according to (B.6), and obtains

$$[1^n] = \sum_{\{n_i m_i = n\}} n! \prod_i \frac{1}{(n_i!)^{m_i} m_i!} \left(\left([1^{n_i - 1}], 1\right) \right)^{m_i}. \tag{C.8}$$

The product $[1^n]$ contains terms of arbitrary Δ_v, starting from n, but here we are interested only in the part with $\Delta_v = n+1$. The part with minimal $\Delta_v = n$ is obtained when all brackets have their minimal $\Delta_v = n_i$ – the part with $\Delta_v = n+1$ is obtained when only one of the brackets has $\Delta_v = n_i + 1$ and all others the minimal one:

$$\begin{aligned} [1^n]_{n+1} = \sum_{\{n_0 + n_i m_i = n\}} & \frac{n!}{n_0!} \left([1^{n_0 - 1}], 1\right)_{n_0+1} \\ & \times \prod_i \frac{1}{(n_i!)^{m_i} m_i!} \left(\left([1^{n_i - 1}], 1\right)_{n_i} \right)^{m_i}. \end{aligned} \tag{C.9}$$

After substituting all the brackets with the corresponding d, and grouping together all the d_n into S_i one obtains the result in (C.7). The expression for $d^{(n)n-1}_{n-1}$ can now be given explicitly in full:

$$(n-1)!\, d^{(n)}_{n-1\,n-1} = (S_{n-2}, 1)_n + \left(S^1_{n-2}, 1\right)_n\,. \tag{C.10}$$

Note that this is again a recursive formula: $d^{(n)}_{n-1\,n-1}$ is expressed in terms of d_m with $m \le n-2$ and $d^{(m)}_{m-1\,m-1}$ with $m \le n-1$.

We now come to the second equation in (C.4): the new object which we have to deal with is $[1^j 2 1^k]_{j+2+k}$. One can express this as follows:

$$\begin{aligned} \left[1^j 2 1^k\right]_{j+2+k} = \sum_{j_1=0}^{j} \binom{j}{j_1} \Bigg[& (S_{j_1}, 2)_{j_1+2}\, S_{j+k-j_1} \\ & + \sum_{k_1=1}^{k} \binom{k}{k_1} \left([1^{j_1} 2 1^{k_1 - 1}], 1\right)_{j_1+k_1+2} S_{j+k-j_1-k_1} \Bigg]. \end{aligned} \tag{C.11}$$

The derivation of this formula follows from the observation that when we split the product on the left-hand side into subsets according to (B.6), only one of the subsets will contain a 2 – the various terms differ by the number of 1s to the left and right of the 2 in the same subset. Moreover, since we are interested here only in the part with minimal Δ_v, the quantity that multiplies the subset with a 2 can be expressed as a S_n. The combinatorial factors are then easily obtained.

We can now insert (C.11) back into (C.4) and get

$$\begin{aligned} n!\, d^{(n)}_{n\,n-1} = {} & (S_{n-2}, 2)_n \\ & + \sum_{j=0}^{n-3} (n-1-j) \sum_{j_1=0}^{j} \binom{j}{j_1} ((S_{j_1}, 2) S_{n-3-j_1}, 1)_n \\ & + \sum_{j=0}^{n-4} (n-1-j) \sum_{j_1=0}^{j} \binom{j}{j_1} \sum_{k_1=1}^{n-3-j} \binom{n-3-j}{k_1} \end{aligned}$$

$$\times \left(\left(\left[1^{j_1}21^{k_1-1}\right],1\right)S_{n-3-j_1-k_1},1\right)_n \,. \quad \text{(C.12)}$$

One can again use (C.11) to simplify further the last term, and one gets

$$\begin{aligned}
n!d^{(n)}_{n\,n-1} &= (S_{n-2},2)_n \\
&+\sum_{j=0}^{n-3}(n-1-j)\sum_{j_1=0}^{j}\binom{j}{j_1}((S_{j_1},2)S_{n-3-j_1},1)_n \\
&+\sum_{j=0}^{n-4}(n-1-j)\sum_{j_1=0}^{j}\binom{j}{j_1}\sum_{k_1=1}^{n-3-j}\binom{n-3-j}{k_1} \\
&\quad\times\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}(((S_{j_2},2)S_{j_1+k_1-1-j_2},1)S_{n-3-j_1-k_1},1)_n \\
&+\sum_{j=0}^{n-5}(n-1-j)\sum_{j_1=0}^{j}\binom{j}{j_1}\sum_{k_1=1}^{n-3-j}\binom{n-3-j}{k_1} \\
&\quad\times\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{k_2=1}^{k_1-1}\binom{k_1-1}{k_2} \qquad \text{(C.13)} \\
&\quad\times\left(\left(\left(\left[1^{j_2}21^{k_2-1}\right],1\right)S_{j_1+k_1-1-j_2-k_2},1\right)S_{n-3-j_1-k_1},1\right)_n .
\end{aligned}$$

One can now use (C.11) as many times as it is needed and fully eliminate the terms containing $[1^j21^k]$. In this manner one obtains an expression for the subleading poles which is fully explicit, again in the sense of a recursive formula: to have leading and subleading poles at $\hbar$-order n one must have already worked out all leading and subleading poles of $\hbar$-order $n' < n$, and insert these as vertices in one- and two-loop diagrams.

D Renormalization of the effective action

In Sect. 5 we have shown that in order to have the sum of 1PR contributions to the generating functional automatically finite, order by order in the loop expansion, one should systematically renormalize the effective actions $\Gamma_n[\phi, J]$ also away from the classical solution. This can always be done because the divergences of Γ_n must be local: to renormalize it is enough to include in the counterterm basis also terms that vanish at the EoM. In this section we discuss in some more details how this can be done, and consider the case of Γ_n, assuming that all Γ_k with $k < n$ have already been renormalized also away from the EoM. This implies that $Z_n^{\rm 1PR}$ is finite, and therefore that $\bar\Gamma_n$ is also finite. We can write a Γ_n which is finite at the EoM as

$$\Gamma_n = \sum_{i=1}^{n}\varepsilon^{-i}\Gamma_{n\,i} + \Gamma_n^f(\varepsilon), \qquad \bar\Gamma_{n\,i} = 0\,, \qquad \text{(D.1)}$$

with $\Gamma_n^f(0)$ finite. Our argument applies to all $\Gamma_{n\,i}$ and to simplify the notation we drop the subscript. Since Γ vanishes at the EoM, we can write it as

$$\Gamma = \sum_n \hat c_n X_r^n S_0^r =: X_r S_0^r\,, \qquad \text{(D.2)}$$

with $S_0^r = 0$ the classical EoM. We are now interested to study its behavior away from the EoM, and can conveniently do this with a Taylor expansion:

$$\phi = \phi_{\rm cl} + \xi \Rightarrow \Gamma = \bar\Gamma^a\xi_a + O\left(\xi^2\right)\,. \qquad \text{(D.3)}$$

We want to ensure that Γ vanishes also away from the EoM, and therefore that

$$\bar\Gamma^a = \bar X_r \Delta^{ra} = 0\,, \qquad \text{(D.4)}$$

which can only be true if $\bar X_r = 0$. This condition can be easily satisfied by properly adjusting the coefficients $\hat c_n$ in front of the counterterms that vanish at the EoM (D.2). X_r, however, may still be different from zero away from the EoM:

$$X_r = X_{rs}S_0^s\,, \qquad \text{(D.5)}$$

which implies

$$\Gamma = \frac{1}{2}\bar\Gamma^{ab}\xi_a\xi_b + O\left(\xi^3\right)\,, \qquad \bar\Gamma^{ab} = \bar X_{rs}\Delta^{ra}\Delta^{sb}\,. \quad \text{(D.6)}$$

$\Gamma^{ab} = 0$ implies $\bar X_{rs} = 0$, which can be obtained by tuning the coefficients of the counterterms that vanish quadratically at the EoM, and in turn this means

$$X_{rs} = X_{rst}S_0^t\,, \qquad \text{(D.7)}$$

and so on. Note that the expansion in ξ of a term that vanishes at the EoM contains powers of the inverse propagator Δ: when inserted in 1PR graphs such vertices will generate local, possibly divergent contributions. By changing the coefficients in front of the EoM terms one can shift local contributions from Γ_n to $Z_n^{\rm 1PR}$.

We stress that the procedure for the renormalization of Γ_n outlined above does not need to go on forever. First of all because at every finite $\hbar$-order only a finite number of conditions have to be imposed; cf. (5.5). Moreover, in any QFT, the EoM must reduce, in a well-defined limit, to the free-field ones:

$$S_0^i = \left(\Box + M^2\right)\phi^i + \sigma^{ij}\phi_j = 0\,. \qquad \text{(D.8)}$$

For dimensional reasons the powers of S_0^i which can be contained in a counterterm of $\hbar$-order n is bounded. For example in a renormalizable theory not more than one power of S_0^i can appear in a counterterm of any $\hbar$-order. In a non-renormalizable theory higher-dimensional interactions are suppressed by powers of an energy scale. This ordering is usually reflected in the $\hbar$-ordering, such that at each order in $\hbar$ only a limited power of S_0^i can appear. In CHPT, e.g., the EoM are of chiral order two, such that at $\hbar$-order one (chiral order four) not more than two powers of S_0^i are allowed: the condition

$$\Gamma_1^i = \Gamma_1^{ij} = 0\,, \qquad \text{(D.9)}$$

ensures that Γ_1 is finite, even away from the classical solution. In general, for Γ_n the chiral counting implies that there are $n+1$ conditions to be imposed:

$$\Gamma_n^{i_1} = \Gamma_n^{i_1 i_2} = \Gamma_n^{i_1\ldots i_{n+1}} = 0\,. \qquad \text{(D.10)}$$

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